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Ground state solution for a kind of Choquard equations with doubly critical exponents and local perturbation

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Abstract: At present, many achievements have been made in the research of Choquard equation. In this paper, some new conclusions about the Choqaurd type problems are obtained through the variational method. It is different from the other Choqaurd equations with critical exponent, in this study, we consider a kind of Choqaurd equations with doubly critical exponents and local perturbation at the same time. The key point of this paper is that Sobolev embedding from workspace to Lebesgue space is not compact. Therefore, it is almost impossible to use conventional methods to obtain the convergence of $(\mathrm{PS})_{\mathrm{c}}$ sequences. In order to eliminate the difficulty, We use the Pohožaev manifold to complete the proof of the existence of solutions for a class of Choquard equations. We also take some new tricks in this equation.

Keywords: Choquard equation; Ground state solution; local perturbation; Pohožaev manifold

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1. Introduction

This article is mainly concerned with the following autonomous Choquard equations:

$$-\Delta u + u = (I_\alpha * |u|^{q_1})\,|u|^{q_1-2}u + (I_\alpha * |u|^{q_2})\,|u|^{q_2-2}u + |u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

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where $N \geq 3$, $\alpha \in (0, N)$,$q_1 = 1 + \frac{\alpha}{N}$, $q_2 = \frac{N+\alpha}{N-2}$, $2 < p < \frac{2N}{N-2}$, I_α is Riesz potential, defined as

$$I_\alpha(x) := |x|^{\alpha-N} A_{(N,\alpha)}, \quad \text{and} \quad A_{(N,\alpha)} = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\alpha/2)\pi^{N/2}2^\alpha}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}. \tag{1.2}$$

Both the lower exponent q_1 and the upper critical exponent q_2 come from the Hardy-Littlewood-Sobolev inequality [1]. Let us review some important physical backgrounds of Choqaurd equation. It seems to originate from Pekar's model of the polaron to describe free electrons in an ionic lattice interact with phonons in 1954 [2]. In a pioneering work, Dr. Choquard developed it into one of the model of a one-component plasma and Lieb has done a lot of work in multiplicity of normalized results [3]. There are also many equations related to Choquard equation, which have extremely rich and complex physical background, such as Hartree, Schrödinger-type evolution equation [4] and so on. For more information on the physical background and details of Choquard equation, we refer readers to literature [5].

If $N > 5$ and $\alpha < N - 4$, Seok [6] showed that the following equation has a nontrivial solution:

$$\begin{cases} \Delta u + u = (I_\alpha * F(u))\, F'(u) \quad \text{in } \mathbb{R}^N; \\ \lim\limits_{x\to\infty} u(x) = 0, \end{cases} \tag{1.3}$$

where $F(u) = \frac{1}{q_1}|u|^{q_1} + \frac{1}{q_2}|u|^{q_2}$. After that, a lot of work has been done to improve (1.3) and some new results have been obtained. Su [7] used Sobolev inequality with Morrey norm to establish the existence of nonnegative solution for (1.3). In order to weaken the condition of N, by the invariant sets of descending flow and perturbation method, Liu, Yang and Chen [8] obtained (1.3) admits infinitely many sign-changing solutions in the case of $N \geq 3$ and $0 < \alpha < N$. When $A_{(N,\alpha)} = 1$ and $F(u) = |u|^{q_1} + |u|^{q_2}$, Lei and Zhang [9] obtained a ground state solution of (1.3). To solve the loss of compactness resulting from critical nonlinearities, they applied the Pohožaev-type identity. This approach will also be used in our paper. For a class of Choquard equations with steep potential well and doubly critical exponents, Li, Li and Tang [10] proved existence and concentration of nonnegative ground state solutions if external variable potential is positive in $\mathbb{R}^N$.

Starting from the autonomous Choquard equation, Li, Ma, Zhang [11] obtained that the nonautonomous equation (excluding the case of critical exponent) still has the ground state solution. In [12], Li and Tang investigated the following equation with upper critical exponent and nonlinear perturbation,

$$-\Delta u + u = (I_\alpha * |u|^{q_2})\, |u|^{q_2-1} + g(u) \quad \text{in}\, \mathbb{R}^N, \tag{1.4}$$

where $g(u)$ is a function of general subcritical growth. And then they obtained that (1.4) has a positive ground state solution. Soon after, Through some mild restrictions on g, Guo, Tang, Zhang and Gao [13] extended the relevant results of nontrivial solutions to (1.4) with variable potential. There are also some nice results on critical exponential growth (or doubly critical exponents) in reference [14, 15, 16, 17, 18]. Motivated by the above papers, especially [9, 11], we shall give the existence of ground state solution for (1.1).

We all know that if u is a solution to (1.1), then u must be the critical point of the energy functional $I : X \triangleq H^1(\mathbb{R}^N) \to \mathbb{R}$, which is defined as

$$I(u) = \frac{1}{2}\int\limits_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int\limits_{\mathbb{R}^N} |u|^2 dx - \frac{1}{2q_1}\mathcal{L}_1(u) - \frac{1}{2q_2}\mathcal{L}_2(u) - \frac{1}{p}\int\limits_{\mathbb{R}^N} |u|^p dx, \tag{1.5}$$

where

$$\mathcal{L}_1 = \int\limits_{\mathbb{R}^N} (I_\alpha * |u|^{q_1})\, |u|^{q_1-1} u dx, \quad \mathcal{L}_2 = \int\limits_{\mathbb{R}^N} (I_\alpha * |u|^{q_2})\, |u|^{q_2-1} u dx. \tag{1.6}$$

Moreover, for any $u, w \in H^1(\mathbb{R}^N)$,

$$\begin{aligned}\langle I'(u), w\rangle = &\int\limits_{\mathbb{R}^N} [\nabla u \cdot \nabla w + uw]\, dx - \int\limits_{\mathbb{R}^N} (I_\alpha * |u|^{q_1})\, |u|^{q_1-2} uw dx - \int\limits_{\mathbb{R}^N} (I_\alpha * |u|^{q_2})\, |u|^{q_2-2} uw dx \\ &- \int\limits_{\mathbb{R}^N} |u|^{p-2} uw dx.\end{aligned} \tag{1.7}$$

In order to overcome the difficulty of proving the subsequence of bounded Palais-Smale (PS) sequence with strong convergence on X, we apply the Pohožaev identity (see [19, 20]). Before presenting the result, we first define a Pohožaev manifold similar to [9]:

$$\mathcal{M} = \{u \in X \setminus \{0\} : \mathcal{P}(u) = 0\}, \tag{1.8}$$

where

$$\mathcal{P}(u) = \frac{N-2}{2}\int\limits_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N}{2}\int\limits_{\mathbb{R}^N} |u|^2 dx - \frac{N}{2}\mathcal{L}_1(u) - \frac{N-2}{2}\mathcal{L}_2 - \frac{N}{p}\int\limits_{\mathbb{R}^N} |u|^q dx. \tag{1.9}$$

Our result is as follows.

Theorem 1.1. *Assume $N \geq 3$, $\alpha \in (0, N)$,$q_1 = \frac{N+\alpha}{N}$, $q_2 = \frac{N+\alpha}{N-2}$ and $2 < p < \frac{2N}{N-2}$, then (1.1) has a ground state solution on $\mathcal{M}$.*

The paper is organized as follows. In Section 2, we prove that I has mountain pass structure and give some necessary lemmas. Finally, we prove that Eq. (1.1) has a ground state solution in Section 3.

The settings of notations in this article are as follows.

- $H^1(\mathbb{R}^N)$ denotes the Sobolev space and the norm and inner product are shown below
$$\|u\| = \langle u, u\rangle^{\frac{1}{2}}, \quad \langle u, w\rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla w + uw)dx, \quad \forall\, u, w \in H^1(\mathbb{R}^N).$$
- $L^s(\mathbb{R}^N)\,(s \in [1, +\infty))$ denotes the Lebesgue space and the norm is $\|u\|_s = \left(\int_{\mathbb{R}^N} |u|^s dx\right)^{1/s}$.
- For any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, $u_t(x) := u(t^{-1}x)$ for $t > 0$.
- C denote different positive constants in different positions.

2. Preliminaries

In this section, we verify the mountain pass structure of I and give some lemmas.

Lemma 2.1. *The functional I is unbounded from below.*

Proof. Let $u \in X$, and $u_t = u(t^{-1}x)$ for $t > 0$. The standard scaling shows that
$$I(u_t) = \frac{t^{N-2}}{2}\|\nabla u\|_2^2 + \frac{t^N}{2}\|u\|_2^2 - \frac{t^{N+\alpha}}{2q_1}\mathcal{L}_1(u) - \frac{t^{N+\alpha}}{2q_2}\mathcal{L}_2(u) - \frac{t^N}{p}\|u\|_p^p. \tag{2.1}$$
It is easy to follow from our assumptions that $I(u_t) \to -\infty$ as $t \to +\infty$. □

Lemma 2.2. *Let a, b, c, d, and e be positive constants, and*
$$h(t) = at^{N-2} + bt^N - ct^{N+\alpha} - dt^{N+\alpha} - et^N$$
Then h has a unique critical point, corresponding to its maximum.

Proof. Different from the direct method in [11, Lemma 4.1], we prove Lemma 2.2 by argument of contradiction. Assume h has at least two positive critical points $t_1 \neq t_2$, noting that $h(0) = 0$, $h(t) \to 0^+$ as $t \to 0^+$ and $h(t) \to -\infty$ as $t \to +\infty$, then we know that h has at least four different critical points, we can assume that
$$0 < t_1 < t_2 < t_3 < +\infty \tag{2.2}$$

such that

$$h(0) = h(t_1) = h(t_2) = h(t_3) = 0.$$

For $t > 0$

$$h'(t) = t^{N-3} h_1(t),$$

where

$$h_1(t) = a(N-2) + bNt^2 - c(N+\alpha)t^{2+\alpha} - d(N+\alpha)t^{2+\alpha} - eNt^2.$$

We know that $h_1(t) = 0$ for $t > 0$ has at least three positive solutions. That is, there exist $t_4 \in (0,\, t_1)$, $t_5 \in (t_1,\, t_2)$ and $t_6 \in (t_2,\, t_3)$ such that

$$h_1(t_4) = h_1(t_5) = h_1(t_6) = 0. \tag{2.3}$$

Since

$$h_1'(t) = t\left[2bN - 2eN - c(N+\alpha)(3+\alpha)t^{\alpha} - d(N+\alpha)(2+\alpha)t^{\alpha}\right], \tag{2.4}$$

we know that $h_1'(t)$ exists at least two solutions, but (2.4) shows that h_1' has at most one solution for $t > 0$. The result contradicts the original hypothesis. Thus, h has a unique positive critical point. □

Lemma 2.3. *Let $u \in X \setminus 0$ be a solution of* (1.1)*, then there exists a path $\gamma \in C([0,1], X)$ such that*

$$\gamma(0) = 0, \quad \gamma\left(\frac{1}{2}\right) = u, \quad I(\gamma(1)) < 0.$$

and

$$I(\gamma(t)) < I(u) \quad for\ any\ t \in [0,1] \setminus \{1/2\}. \tag{2.5}$$

Proof. We first define the path $\bar{\gamma} : [0,\infty) \to X$ as

$$\bar{\gamma} = \begin{cases} u(t^{-1}x), & t > 0, \\ 0, & t = 0. \end{cases}$$

Then from (2.1), we have

$$I(\bar{\gamma}(t)) = \frac{t^{N-2}}{2}\|\nabla u\|_2^2 + \frac{t^N}{2}\|u\|_2^2 - \frac{t^{N+\alpha}}{2q_1}\mathcal{L}_1(u) - \frac{t^{N+\alpha}}{2q_2}\mathcal{L}_2(u) - \frac{t^N}{p}\|u\|_p^p. \tag{2.6}$$

It is obvious that $\bar{\gamma}$ is continuous for $t>0$. When $t=0$, $\bar{\gamma}$ is also continuous. Since

$$\lim_{t\to 0^+}\left[\int\limits_{\mathbb{R}^N}|\nabla\bar{\gamma}(t)|^2+\int\limits_{\mathbb{R}^N}|\bar{\gamma}(t)|^2dx\right]=\lim_{t\to 0^+}\left[t^{N-2}\int\limits_{\mathbb{R}^N}\nabla|u|^2dx+t^N\int\limits_{\mathbb{R}^N}|u|^2dx\right]=0.$$

From (1.9), we have

$$\frac{1}{p}\int\limits_{\mathbb{R}^N}|u|^pdx=\frac{N-2}{2N}\int\limits_{\mathbb{R}^N}|\nabla u|^2dx+\frac{1}{2}\int\limits_{\mathbb{R}^N}|u|^2dx-\frac{1}{2}\mathcal{L}_1(u)-\frac{N-2}{2N}\mathcal{L}_2(u). \tag{2.7}$$

Substituting (2.7) into (2.6), for $t>0$, we obtain

$$\begin{aligned}I(\bar{\gamma}(t))&=\left(\frac{1}{2}t^{N-2}-\frac{N-2}{2N}t^N\right)\|\nabla u\|_2^2+\left(\frac{1}{2}t^N-\frac{1}{2q_1}t^{N+\alpha}\right)\mathcal{L}_1(u)\\&\quad+\left(\frac{N-2}{2N}t^N-\frac{1}{2q_2}t^{N+\alpha}\right)\mathcal{L}_2(u).\end{aligned} \tag{2.8}$$

By simple calculation, we have

$$\left.\frac{dI(\bar{\gamma}(t))}{dt}\right|_{t=1}=0.$$

It is clear that $t=1$ is a critical point of $I(\bar{\gamma}(t))$.

In view of Lemma 2.2 and (2.6), $I(\bar{\gamma}(t))$ has a unique critical point $t=1$ corresponds to its maximum. And we also know $\lim_{t\to\infty}I(\bar{\gamma}(t))=-\infty$. Therefore, the path γ can be used by a suitable change of variable. □

By Lemma 2.2 and 2.3, the maximum of $I(u_t)$ is achieved at $t=1$ and $I'(u_t)=0$ as $t=1$. Therefore, $\mathcal{M}\neq\emptyset$. Next, we give some properties of $\mathcal{M}$.

Lemma 2.4. *$\mathcal{M}$ is a C^1-manifold and every critical point of I in $\mathcal{M}$ is a critical point of I.*

Proof. Let's achieve it in four steps.

Step 1. We first give the important characteristic $0\notin\partial\mathcal{M}$.

For $u\in X\setminus\{0\}$, by the Hardy-Littlewood-Sobolev inequality, we have

$$I(u)\geq\frac{1}{2}\|u\|^2-C\|u\|^{2q_1}-C\|u\|^{2q_2}-C\|u\|^p$$

for some $C>0$. There must be a pair of constants ρ, $\varepsilon>0$ (ε enough small), such that $I(u)\geq\rho$ for $\|u\|=\varepsilon$. Therefore, $0\notin\partial\mathcal{M}$.

Step 2. By (1.9), we have $\inf_{\mathcal{M}}I>0$.

For any $u \in \mathcal{M}$ and from (1.5) and (1.9), we have

$$\begin{aligned} I(u) =&\frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \frac{1}{2q_1}\mathcal{L}_1(u) - \frac{1}{2q_2}\mathcal{L}_2(u) - \left(\frac{N-2}{2N}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \frac{1}{2}\mathcal{L}_1(u)\right. \\ &\left. - \frac{N-2}{2N}\mathcal{L}_2(u)\right) \\ =&\frac{1}{N}\|\nabla u\|_2^2 + \left(\frac{1}{2} - \frac{1}{2q_1}\right)\mathcal{L}_1(u) + \left(\frac{N-2}{2N} - \frac{N-2}{2(N+\alpha)}\right)\mathcal{L}_2(u) > 0 \end{aligned} \tag{2.9}$$

Step 3. $\mathcal{M}$ is a C^1-manifold.

According to the implicit function theorem, we just need to prove $\mathcal{P}'(u) \neq 0$ for every $u \in \mathcal{M}$. As usual, arguing by contradiction, we suppose that $\mathcal{P}'(u) = 0$ for some $u \in \mathcal{M}$.

Set

$$\beta = \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad \omega = \int_{\mathbb{R}^N} |u|^2 dx, \quad \sigma = \int_{\mathbb{R}^N} |u|^p dx. \tag{2.10}$$

Then β, ω, $\mathcal{L}_1(u)$, $\mathcal{L}_2(u)$ and σ are positive. From $\mathcal{P}(u) = 0$ and $\langle \mathcal{P}'(u), u\rangle = 0$, we have

$$\begin{cases} \frac{N-2}{2}\beta + \frac{N}{2}\omega - \frac{N}{2}\mathcal{L}_1(u) - \frac{N-2}{2}\mathcal{L}_2(u) - \frac{N}{p}\sigma = 0; \\ (N-2)\beta + N\omega - (N+\alpha)\mathcal{L}_1(u) - (N+\alpha)\mathcal{L}_2(u) - N\sigma = 0. \end{cases} \tag{2.11}$$

By direct calculation, (2.11) shows that

$$\alpha\mathcal{L}_1(u) + (\alpha+2)\mathcal{L}_2(u) + \left(1 - \frac{2}{p}\right)N\sigma = 0. \tag{2.12}$$

Since $p > 2$, (2.12) shows a contradiction. Therefore, $\mathcal{M}$ is a C^1-manifold.

Step 4. Finally, we prove that every critical point of I on $\mathcal{M}$ is critical point of I in X.

If u is a critical point of I on $\mathcal{M}$, there must be a constant $\lambda \in \mathbb{R}$ called Lagrange multiplier such that $I'(u) = \lambda\mathcal{P}'(u)$. We can obtain the following equation

$$\begin{aligned} -(\lambda(N-2)-1)\Delta u + (N\lambda - 1)u =& \left[(N+\alpha)\lambda - 1\right](I_\alpha * |u|^{q_1})|u|^{q_1-2}u + (N\lambda-1)|u|^{p-2}u \\ &+ \left[(N+\alpha)\lambda - 1\right](I_\alpha * |u|^{q_2})|u|^{q_2-2}u. \end{aligned} \tag{2.13}$$

We just need to prove that $\lambda = 0$. The Pohožaev identity of (2.13) is

$$\begin{aligned} \frac{N-2}{2}((N-2)\lambda-1)\beta + \frac{N}{2}(N\lambda-1)\omega - \frac{N}{2}((N+\alpha)\lambda-1)\mathcal{L}_1(u) - \frac{N-2}{2}((N+\alpha)\lambda-1)\mathcal{L}_2(u) \\ - \frac{N}{p}(N\lambda-1)\sigma = 0. \end{aligned} \tag{2.14}$$

From $\mathcal{P}(u) = 0$ and (2.14), we have

$$\lambda\left(\frac{(N-2)^2}{2}\beta + \frac{N^2}{2}\omega - \frac{N(N+\alpha)}{2}\mathcal{L}_1(u) - \frac{(N-2)(N+\alpha)}{2}\mathcal{L}_2(u) - \frac{N^2}{p}\sigma\right) = 0. \tag{2.15}$$

If $\lambda \neq 0$, (2.15) and $\mathcal{P}(u) = 0$ give that

$$0 = \left(\frac{N(N-2)}{2} - \frac{(N-2)^2}{2}\right)\beta + \left(\frac{N(N+\alpha)}{2} - \frac{N^2}{2}\right)\mathcal{L}_1(u) + \left(\frac{(N-2)(N+\alpha)}{2} - \frac{N(N+\alpha)}{2}\right)\mathcal{L}_2(u). \tag{2.16}$$

(2.16) shows a contradicts. Hence $\lambda = 0$. Consequently, our expected results are obtained by four steps. □

3. Proof of Theorem 1.1.

In this section, we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Assume that $\{u_n\}$ be a minimizing sequence of I in $\mathcal{M}$, so $I(u_n) \to \inf_{\mathcal{M}} I$ as $n \to \infty$. In fact, $\{u_n\}$ is a (PS) sequence of I. Here we give some necessary proofs. By Ekeland variational principle, there exist $\{u_n\} \subset \mathcal{M}$ and $\{\lambda_n\} \subset \mathbb{R}$ such that

$$I(u_n) \to \inf_{\mathcal{M}} I, \quad I'(u_n) - \lambda_n \mathcal{P}'(u_n) \to 0 \quad as \quad n \to \infty. \tag{3.1}$$

Similar to the Step 4 in Lemma 2.4, we can deduce that $\lim_{n\to\infty} \lambda_n = 0$. Then $\{u_n\} \subset \mathcal{M}$ is a (PS) sequence of I. Thus $I(u_n) \to \inf_{\mathcal{M}}$, $I'(u_n) \to 0$ as $n \to \infty$.

By (2.9), there holds

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \left(\frac{1}{2} - \frac{1}{2q_1}\right)\mathcal{L}_1(u_n) + \left(\frac{N-2}{2N} - \frac{N-2}{2(N+\alpha)}\right)\mathcal{L}_2(u_n) \to \inf_{\mathcal{M}} I \ (n \to \infty).$$

This means that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \le C < +\infty, \quad \mathcal{L}_1(u_n) \le C < +\infty, \quad \mathcal{L}_2(u_n) \le C < +\infty \tag{3.2}$$

for some $C > 0$.

It follows from $\mathcal{P}(u_n) = 0$ and $\langle \mathcal{P}'(u_n), u_n\rangle = 0$ that

$$(N-2)\left(\frac{p}{2} - 1\right)\|\nabla u\|_2^2 + N\left(\frac{p}{2} - 1\right)\|u\|_2^2 = \left[\frac{(N-2)p}{2} - N - \alpha\right]\mathcal{L}_2(u)$$

$$+\left(\frac{Np}{2}-N-\alpha\right)\mathcal{L}_1(u). \tag{3.3}$$

From (3.2) and (3.3), we can deduce that

$$\int\limits_{\mathbb{R}^N}|u_n|^2dx\le C<\infty\quad\text{for some } C>0.$$

Thus, $\{u_n\}$ is bounded in X.

Since $\{u_n\}$ is bounded in X, there exists a subsequence of u_n denoted by itself such that $u_n\rightharpoonup v$ weakly in X as $n\to\infty$. It follows from $I'(u_n)\to 0\,(n\to\infty)$ that

$$\int\limits_{\mathbb{R}^N}[\nabla v\nabla w+vw]dx=A_{(\alpha,N)}\int\limits_{\mathbb{R}^N\times\mathbb{R}^N}\left(\frac{|v(x)|^{q_1}|v(y)|^{q_1-1}w(y)}{|x-y|^{N-\alpha}}+\frac{|v(x)|^{q_2}|v(y)|^{q_2-1}w(y)}{|x-y|^{N-\alpha}}\right)dxdy$$
$$+\int\limits_{\mathbb{R}^N}|v|^{p-1}wdx\qquad\text{for } w\in C_0^\infty(\mathbb{R}^N). \tag{3.4}$$

Then v is a critical point for I. Therefore,

$$\langle I'(v),v\rangle=0\qquad\text{and}\qquad\mathcal{P}(v)=0,$$

which implies that $v\in\mathcal{M}$. By [19, Lemma 2.4], denote $\hat{u}_n:=u_n-v$, then $\hat{u}_n\rightharpoonup 0$ in X as $n\to\infty$, we have

$$\int\limits_{\mathbb{R}^N}|\hat{u}_n|^z dx=\int\limits_{\mathbb{R}^N}|u_n|^z dx-\int\limits_{\mathbb{R}^N}|v|^z dx+o(1),\quad(z=2,p), \tag{3.5}$$

$$\int\limits_{\mathbb{R}^N}|\nabla\hat{u}_n|^2=\int\limits_{\mathbb{R}^N}|\nabla u_n|^2dx-\int\limits_{\mathbb{R}^N}|\nabla v|^2dx+o(1), \tag{3.6}$$

$$\mathcal{L}_1(\hat{u}_n)=\mathcal{L}_1(u_n)-\mathcal{L}_1(v)+o(1),\quad\mathcal{L}_2(\hat{u}_n)=\mathcal{L}_2(u_n)-\mathcal{L}_2(v)+o(1) \tag{3.7}$$

as $n\to\infty$. Hence, from (3.5)-(3.7) and $\mathcal{P}(v)=0$, we have

$$\mathcal{P}(u_n)=\frac{N-2}{2}\int\limits_{\mathbb{R}^N}|\nabla\hat{u}_n|^2dx+\frac{N}{2}\int\limits_{\mathbb{R}^N}|\hat{u}_n|^2dx-\frac{N}{2}\mathcal{L}_1(\hat{u}_n)-\frac{N-2}{2}\mathcal{L}_2(\hat{u}_n)-\frac{N}{p}\int\limits_{\mathbb{R}^N}|\hat{u}_n|^q dx+o(1)$$

when $n\to\infty$. Since $\mathcal{P}(u_n)\to 0$ as $n\to\infty$, we have

$$\mathcal{P}(\hat{u}_n)=o(1). \tag{3.8}$$

From (3.5)-(3.7), we have

$$I(u_n) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla \hat{u}_n|^2 + |\hat{u}_n|^2\right) dx - \frac{1}{2q_1}\mathcal{L}_1(\hat{u}_n) - \frac{1}{2q_2}\mathcal{L}_2(\hat{u}_n) - \frac{1}{p}\int_{\mathbb{R}^N} |\hat{u}_n|^p dx + I(v) + o(1). \quad (3.9)$$

Employing (3.8) to (3.9), we obtain

$$\begin{aligned} I(u_n) =& \frac{1}{N}\int_{\mathbb{R}^N} |\nabla \hat{u}_n|^2 dx + \left(\frac{1}{2} - \frac{1}{2q_1}\right)\mathcal{L}_1(\hat{u}_n) + \left(\frac{N-2}{2N} - \frac{1}{2q_2}\right)\mathcal{L}_2(\hat{u}_n) + I(v) + o(1) \\ \geq& I(v) + o(1) \end{aligned}$$

as $n \to \infty$. Then, we have $I(v) \leq \inf_{\mathcal{M}} I$. Since $v \in \mathcal{M}$, thus $I(v) \geq \inf_{\mathcal{M}} I$. Hence, $I(v) = \inf_{\mathcal{M}} I$. It follows from the Step 2 in Lemma 2.4 that $v \neq 0$. This shows that v is a ground state solution of (1.1). The proof is complete. □

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Declarations

Availability of data and materials

Data sharing not applicable to this article as no data were generated or analysed during the current study.

Competing interests

The authors declare that they has no competing interests.

Authors' contributions

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